

GENERAL ELECTRIC-MAGNETIC DECOMPOSITION OF FIELDS, POSITIVITY AND RAINICH-LIKE CONDITIONS

JOSÉ M.M. SENOVILLA

*Departamento de Física Teórica, Universidad del País Vasco
Apartado 644, 48080 Bilbao, SPAIN
E-mail: wtpmasej@lg.ehu.es*

We show how to generalize the classical electric-magnetic decomposition of the Maxwell or the Weyl tensors to arbitrary fields described by tensors of any rank in general n -dimensional spacetimes of Lorentzian signature. The properties and applications of this decomposition are reviewed. In particular, the definition of tensors quadratic in the original fields and with important positivity properties is given. These tensors are usually called “super-energy” (s-e) tensors, they include the traditional energy-momentum, Bel and Bel-Robinson tensors, and satisfy the so-called Dominant Property, which is a straightforward generalization of the classical dominant energy condition satisfied by well-behaved energy-momentum tensors. We prove that, in fact, any tensor satisfying the dominant property can be decomposed as a finite sum of the s-e tensors. Some remarks about the conservation laws derivable from s-e tensors, with some explicit examples, are presented. Finally, we will show how our results can be used to provide adequate generalizations of the Rainich conditions in general dimension and for any physical field.

1 Introduction

A great deal of theoretical physics is devoted to the *unification* of concepts. Two rather early and outstanding examples of physical unifications are given by the concepts of

<u>ELECTRO-MAGNETISM</u>	and	<u>SPACE-TIME</u>
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which respectively unify the electric field with the magnetic field, and the space and time measurements. Of course, these two unifications are obviously related, and I shall try to maintain the coherence by underlying the corresponding concepts (so that ‘electric’ is related to ‘time’). Thus, for instance, the electromagnetic field is simply the spacetime unification of the classical electric field \vec{E} with the classical magnetic induction \vec{H} .

Nevertheless, it is very important to keep in mind that we do not measure “spacetime intervals”, nor “electromagnetic strengths”. Actually, in labs we can only measure

Space distances	<u>Time</u> intervals
<u>Electric</u> field	Magnetic field

by using rods, clocks, electrometers, Gaussmeters or fluxmeters. This means that we must know how to do the *splitting* of the unified concepts. So we need to define how to do the followign breakings

$$\text{spacetime} \left\{ \begin{array}{l} \text{space} \\ \text{time} \end{array} \right\}, \quad \text{electromagnetic} \left\{ \begin{array}{l} \text{electric} \\ \text{magnetic} \end{array} \right\}$$

which, naturally, are intimately related and, more importantly, depend on the observer. Let us remind that an observer is defined, in General Relativity and similar geometrical theories, by a unit timelike vector field \vec{u} . In the domain of validity of the observer described by \vec{u} one has that

1. The space/time splitting leads to the theory of reference frames. The vector field \vec{u} defines a congruence of timelike curves \mathcal{C} so that, for this observer, the infinitesimal variations of time T are represented somehow by the 1-form $u_\mu \propto \delta_\mu T$, while the corresponding space is defined by the quotient V_n/\mathcal{C} of the manifold by the congruence. Let us remark that, according to some authors^{1,2}, and in my opinion too, the metric structure of this space is debatable, so that a unequivocal way to measure distances (without using light rays) is not clearly defined.
2. The electric/magnetic decomposition has not been pursued in full generality until recently^{3,4,5}. This general decomposition, for any given tensor, will be presented and briefly analized here in any spacetime of *arbitrary dimension* n .

The only pre-requisite we will need in order to achieve the electric/magnetic decomposition is the existence of a metric g with Lorentzian signature (i.e., the possibility of defining time), which we will take to be $(-, +, \dots, +)$. Apart from that, our considerations are completey general and therefore applicable to almost any available geometric physical theory.

1.1 The electromagnetic field as guide ...

Let us take, for a moment, the typical case of an electromagnetic field^a $F_{\mu\nu} = F_{[\mu\nu]}$ in a 4-dimensional Lorentzian manifold (V_4, g) . The electric $\vec{E}(\vec{u})$ and the magnetic $\vec{H}(\vec{u})$ fields relative to the observer \vec{u} are defined at any point by

$$E_\mu(\vec{u}) \equiv F_{\mu\nu} u^\nu, \quad H_\mu(\vec{u}) \equiv \overset{*}{F}_{\mu\nu} u^\nu$$

respectively, where $\overset{*}{F}_{\mu\nu} \equiv (1/2)\eta_{\mu\nu\rho\sigma} F^{\rho\sigma}$ is the Hodge dual of $F_{\mu\nu}$ constructed by using the volume element 4-form η . Obviously we have that

$$E_\mu u^\mu = H_\mu u^\mu = 0$$

^aI use the standard square and round brackets to denote the usual (anty-) symmetrization.

so that the electric and magnetic fields are *spatial vectors* relative to the observer \vec{u} . This implies that $\vec{E}(\vec{u})$ and $\vec{H}(\vec{u})$ have 3 independent components each, which altogether add up to the 6 independent components of the 2-form $F_{\mu\nu}$ in 4 dimensions. These vectors allow to define the following classes of electromagnetic fields

$$F_{\mu\nu} \text{ is called } \begin{cases} \text{purely electric} & \text{if } \exists \vec{u} \text{ such that } \vec{H}(\vec{u}) = \vec{0} \\ \text{purely magnetic} & \text{if } \exists \vec{u} \text{ such that } \vec{E}(\vec{u}) = \vec{0} \\ \text{null} & \text{if } F_{\mu\nu} \overset{*}{F}{}^\nu{}_\rho = 0 \text{ and } F_{\mu\nu} F^{\mu\nu} = 0 \end{cases}$$

Mind, however, that not all electromagnetic fields belong to one of these three classes. In all three types the condition $E_\mu H^\mu = 0$ holds, as this is an invariant: $\forall \vec{u}, 2E_\mu H^\mu = F_{\mu\nu} \overset{*}{F}{}^{\mu\nu}$. The other invariant $E_\mu E^\mu - H_\mu H^\mu = -F_{\mu\nu} F^{\mu\nu}$ selects the particular type according to sign.

The energy-momentum tensor of the electromagnetic field in 4 dimensions is defined by

$$\tau_{\mu\nu} \equiv \frac{1}{2} \left(F_{\mu\rho} F_\nu{}^\rho + \overset{*}{F}{}_{\mu\rho} \overset{*}{F}{}_\nu{}^\rho \right), \quad (1)$$

and satisfies

$$\tau_{\mu\nu} = \tau_{(\mu\nu)}, \quad \tau_\mu{}^\mu = 0, \quad \tau_{\mu\rho} \tau^\rho{}_\nu \propto g_{\mu\nu}. \quad (2)$$

Conditions (2) are the basis of the so-called Rainich theory, to be considered in more generality in sections 4 and 5. By using Maxwell equations, $\nabla_{[\rho} F_{\mu\nu]} = 0$, $\nabla_\rho F^{\rho\mu} = -j^\mu$ where \vec{j} is the 4-current, one derives $\nabla_\rho \tau^{\rho\mu} = F^{\mu\nu} j_\nu$ from where

$$\vec{j} = \vec{0} \implies \nabla_\rho \tau^{\rho\mu} = 0.$$

If there is a (conformal) Killing vector $\vec{\zeta}$, then the last expression leads to

$$\nabla_\rho J^\rho = 0, \quad J^\rho(F; \zeta) \equiv \tau^{\rho\mu} \zeta_\mu$$

so that the *divergence-free vector* (also called current) \vec{J} can be used to construct conserved quantities via Gauss theorem.

The energy density of the electromagnetic field with respect to the observer \vec{u} is defined by

$$\Omega(\vec{u}) \equiv \tau_{\mu\nu} u^\mu u^\nu = \frac{1}{2} (E_\mu E^\mu + H_\mu H^\mu) \quad (3)$$

from where we immediately deduce $\Omega(\vec{u}) \geq 0$ for all timelike \vec{u} and

$$\{\exists \vec{u} \text{ such that } \Omega(\vec{u}) = 0\} \iff \tau_{\mu\nu} = 0 \iff F_{\mu\nu} = 0.$$

Actually, $\tau_{\mu\nu}$ satisfies a stronger condition, usually called the dominant energy condition,^{6,7} given by

$$\tau_{\mu\nu}v^\mu w^\nu \geq 0 \quad \text{for all causal future-pointing } \vec{v}, \vec{w} \quad (4)$$

which is equivalent to saying that the flux energy vector $p^\mu(\vec{u}) \equiv -\tau^\mu{}_\nu u^\nu$ is always causal and future pointing for all possible observers.

1.2 ... and the curvature tensors as inspiration.

In General Relativity, there is another well-known ‘E-H’ decomposition, namely, that of the Weyl tensor, introduced in particular coordinates in⁸ and then in general in⁹, see also^{10,11}. In n dimensions, the Weyl tensor reads

$$C_{\alpha\beta,\lambda\mu} = R_{\alpha\beta,\lambda\mu} - \frac{2}{n-2} (R_{\alpha[\lambda}g_{\mu]\beta} - R_{\beta[\lambda}g_{\mu]\alpha}) + \frac{2R}{(n-1)(n-2)}g_{\alpha[\lambda}g_{\mu]\beta}$$

where $R_{\alpha\beta\lambda\mu}$ is the Riemann tensor, $R_{\alpha\lambda} \equiv R^\rho{}_{\alpha\rho\lambda}$ is the Ricci tensor and $R \equiv R^\rho{}_\rho$ is the scalar curvature, and it has the same symmetry properties as the Riemann tensor but is also traceless

$$C_{\alpha\beta\lambda\mu} = C_{[\alpha\beta][\lambda\mu]}, \quad C_{\alpha[\beta\lambda\mu]} = 0, \quad C^\rho{}_{\beta\rho\mu} = 0.$$

Due to the Lanczos identity, see e.g.⁵, the Weyl tensor has only one independent Hodge dual in 4 dimensions, given by ${}^*C_{\alpha\beta\mu\nu} \equiv (1/2)\eta_{\mu\nu\rho\sigma}C_{\alpha\beta}{}^{\rho\sigma} = (1/2)\eta_{\alpha\beta\rho\sigma}C^{\rho\sigma}{}_{\mu\nu}$. Therefore, one can define the ‘electric’ and ‘magnetic’ parts of the Weyl tensor with regard to \vec{u} respectively as (in $n = 4$!)

$$E_{\alpha\lambda}(\vec{u}) \equiv C_{\alpha\beta\lambda\mu}u^\beta u^\mu, \quad E_{\alpha\lambda} = E_{\lambda\alpha}, \quad E^\mu{}_\mu = 0, \quad u^\alpha E_{\alpha\lambda} = 0, \quad (5)$$

$$H_{\alpha\lambda}(\vec{u}) \equiv {}^*C_{\alpha\beta\lambda\mu}u^\beta u^\mu, \quad H_{\alpha\lambda} = H_{\lambda\alpha}, \quad H^\mu{}_\mu = 0, \quad u^\alpha H_{\alpha\lambda} = 0, \quad (6)$$

so that they are spatial symmetric traceless tensors. Each of them has 5 independent components adding up to the 10 components of $C_{\alpha\beta\lambda\mu}$, see^{5,11,12} for more details. Less known but equally old is the similar decomposition of the full Riemann tensor, also introduced in⁹, and given by the following *four* 2-index tensors (in $n = 4$!)

$$\begin{aligned} Y_{\alpha\lambda}(\vec{u}) &\equiv R_{\alpha\beta\lambda\mu}u^\beta u^\mu, & X_{\alpha\lambda}(\vec{u}) &\equiv {}^*R{}_{\alpha\beta\lambda\mu}u^\beta u^\mu, \\ Z_{\alpha\lambda}(\vec{u}) &\equiv {}^*R_{\alpha\beta\lambda\mu}u^\beta u^\mu, & Z'_{\alpha\lambda}(\vec{u}) &\equiv R{}_{\alpha\beta\lambda\mu}u^\beta u^\mu \end{aligned}$$

where the place of the $*$ indicates the pair of skew indices which are Hodge-dualized, see^{9,10,13}. Taking into account the symmetry properties of $R_{\alpha\beta\lambda\mu}$,

they satisfy^{9,10,13}

$$\begin{aligned} X_{\alpha\lambda} &= X_{\lambda\alpha}, & u^\alpha X_{\alpha\lambda} &= 0, & Y_{\alpha\lambda} &= Y_{\lambda\alpha}, & u^\alpha Y_{\alpha\lambda} &= 0, \\ Z_{\alpha\lambda} &= Z'_{\lambda\alpha}, & u^\alpha Z_{\alpha\lambda} &= 0, & u^\alpha Z'_{\alpha\lambda} &= 0, & Z_\alpha^\alpha &= Z'^\alpha_\alpha = 0 \end{aligned}$$

so that $X_{\alpha\lambda}$ and $Y_{\alpha\lambda}$ have 6 independent components each, while $Z_{\alpha\lambda}$ (or equivalently $Z'_{\alpha\lambda}$) has the remaining 8 independent components of the Riemann tensor. They wholly determine the Riemann tensor.

It is important to stress here that the above properties and decompositions depend *crucially on the dimension* of the spacetime, and many of the above simple things are no longer true in general dimension n . As perhaps a surprising example let us remark that the ‘E-H’ decomposition of the Weyl tensor has in general *four* different spatial tensors (as that of the Riemann tensor above). All this will be analyzed in the next section and clarified in subsection 2.2.

2 Tensors as r -folded forms: general E-H decompositions.

In order to define a fully general ‘electric-magnetic’ decomposition of any m -covariant tensor $t_{\mu_1 \dots \mu_m}$, the key idea is to split its indices into antisymmetric blocks (say r in total) denoted generically by $[n_\Upsilon]$, where n_Υ is the number of antisymmetric indices in the block and $\Upsilon = 1, \dots, r$, see^{3,5}. In this way, we look at $t_{\mu_1 \dots \mu_m}$ as an r -fold (n_1, \dots, n_r) -form, where obviously $n_1 + \dots + n_r = m$, and use the notation $t_{[n_1], \dots, [n_r]}$ for the tensor.

Then, one constructs all the duals by using the Hodge $*$ with the volument element $\eta_{\mu_1 \dots \mu_n}$ acting on each of the $[n_\Upsilon]$ blocks, obtaining a total of 2^r different tensors, each of which is an r -fold form (except when $n_\Upsilon = n$ for some Υ). I define the canonical ‘E-H’ decomposition of $t_{\mu_1 \dots \mu_m}$ relative to \vec{u} by contracting each of these duals on all their r blocks with $\vec{u}^{\vec{B},5}$: whenever \vec{u} is contracted with a ‘starred’ block $[n - n_\Upsilon]^*$, we get a ‘magnetic part’ in that block, and an ‘electric part’ otherwise. Thus, the electric-magnetic parts are (generically) r -fold forms which can be denoted by

$$\begin{aligned} & (\vec{u} \underbrace{EE \dots E}_r)_{[n_1-1], [n_2-1], \dots, [n_r-1]}, \quad (\vec{u} \underbrace{HE \dots E}_{r-1})_{[n-n_1-1], [n_2-1], \dots, [n_r-1]}, \quad \dots \\ & (\vec{u} \underbrace{E \dots EH}_{r-1})_{[n_1], \dots, [n_{r-1}-1], [n-n_r-1]}, \quad (\vec{u} \underbrace{HH E \dots E}_{r-2})_{[n-n_1-1], [n-n_2-1], \dots, [n_r-1]}, \\ & \dots, \quad (\vec{u} \underbrace{E \dots EHH}_{r-2})_{[n_1-1], \dots, [n-n_{r-1}-1], [n-n_r-1]}, \quad \dots \\ & \dots, \dots, \quad (\vec{u} \underbrace{HHH \dots H}_r)_{[n-n_1-1], [n-n_2-1], \dots, [n-n_r-1]}, \end{aligned}$$

where, for instance,

$$\begin{aligned}
(\overset{t}{\vec{u}} \underbrace{E E \dots E}_r)_{\mu_2 \dots \mu_{n_1}, \dots, \rho_2 \dots \rho_{n_r}} &\equiv \tilde{t}_{\mu_1 \mu_2 \dots \mu_{n_1}, \dots, \rho_1 \rho_2 \dots \rho_{n_r}} u^{\mu_1} \dots u^{\rho_1}, \\
(\overset{t}{\vec{u}} \underbrace{H E \dots E}_{r-1})_{\mu_{n_1+2} \dots \mu_n, \dots, \rho_2 \dots \rho_{n_r}} &\equiv \tilde{t}_{\mu_{n_1+1} \mu_{n_1+2} \dots \mu_n, \dots, \rho_1 \rho_2 \dots \rho_{n_r}}^* u^{\mu_{n_1+1}} \dots u^{\rho_1} = \\
&= \eta_{\mu_1 \mu_2 \dots \mu_n} \tilde{t}^{\mu_1 \mu_2 \dots \mu_{n_1}, \dots, \rho_1 \rho_2 \dots \rho_{n_r}} u^{\mu_{n_1+1}} \dots u^{\rho_1}
\end{aligned}$$

and so on. Here, $\tilde{t}_{[n_1], \dots, [n_r]}$ denotes the tensor obtained from $t_{[n_1], \dots, [n_r]}$ by permutting the indices such that they are in the order given by $[n_1] \dots [n_r]$. In general, there are 2^r independent E-H parts, they are *spatial* relative to \vec{u} in the sense that they are orthogonal to \vec{u} in any index, and all of them determine $t_{\mu_1 \dots \mu_m}$ completely. Besides, $t_{\mu_1 \dots \mu_m}$ vanishes iff all its E-H parts do.

A special case of relevance is that of *decomposable* r -fold forms: $t_{[n_1], \dots, [n_r]}$ is said to be decomposable if there are r forms $\Omega_{\mu_1 \dots \mu_{n_\Upsilon}}^{(\Upsilon)} = \Omega_{[\mu_1 \dots \mu_{n_\Upsilon}]}^{(\Upsilon)}$ ($\Upsilon = 1 \dots r$) such that

$$\tilde{t}_{[n_1] \dots [n_r]} = \left(\Omega^{(1)} \otimes \dots \otimes \Omega^{(r)} \right)_{[n_1] \dots [n_r]}. \quad (7)$$

Obviously, any r -fold form is a sum of decomposable ones.

2.1 Single blocks or single p -forms

In order to see what the ‘E-H’ decomposition accomplishes, let us concentrate on a single block $[n_\Upsilon]$ with $n_\Upsilon = p$ indices or, for the sake of clarity and simplicity, on a single p -form $\Sigma_{\mu_1 \dots \mu_p} = \Sigma_{[\mu_1 \dots \mu_p]}$. In general, Σ has $\binom{n}{p}$ independent components. The electric part of Σ with respect to \vec{u} is the $(p-1)$ -form

$$(\overset{\Sigma}{\vec{u}} E)_{\mu_2 \dots \mu_p} = (\overset{\Sigma}{\vec{u}} E)_{[\mu_2 \dots \mu_p]} \equiv \Sigma_{\mu_1 \mu_2 \dots \mu_p} u^{\mu_1}, \quad (\overset{\Sigma}{\vec{u}} E)_{\mu_2 \dots \mu_p} u^{\mu_2} = 0.$$

Obviously, $(\overset{\Sigma}{\vec{u}} E)$ has $\binom{n-1}{p-1}$ independent components, which correspond to those with a ‘zero’ among the components $\Sigma_{\mu_1 \dots \mu_p}$ in any orthonormal basis $\{\vec{e}_\mu\}$ with $\vec{e}_0 = \vec{u}$, that is, to $\Sigma_{0i_2 \dots i_p}$ (we use italic lower-case indices $i, j, \dots = 1, \dots, n-1$). Similarly, the magnetic part of Σ with respect to \vec{u} is the $(n-p-1)$ -form

$$(\overset{\Sigma}{\vec{u}} H)_{\mu_2 \dots \mu_{n-p}} = (\overset{\Sigma}{\vec{u}} H)_{[\mu_2 \dots \mu_{n-p}]} \equiv \Sigma_{\mu_1 \mu_2 \dots \mu_{n-p}}^* u^{\mu_1}, \quad (\overset{\Sigma}{\vec{u}} H)_{\mu_2 \dots \mu_{n-p}} u^{\mu_2} = 0.$$

$(\frac{\Sigma}{\vec{u}}\mathbf{H})$ has $\binom{n-1}{n-p-1} = \binom{n-1}{p}$ independent components, which correspond to those without ‘zeros’ in the basis above, that is, to $\Sigma_{i_1 \dots i_p}$. The sum of the components of the electric part plus the components of the magnetic part gives

$$\binom{n-1}{p-1} + \binom{n-1}{p} = \binom{n}{p}$$

which is the total number of components of Σ . The single p -form Σ can be expressed in terms of its electric and magnetic parts as⁵

$$\begin{aligned} \Sigma_{\mu_1 \dots \mu_p} &= -p u_{[\mu_1} (\frac{\Sigma}{\vec{u}}E)_{\mu_2 \dots \mu_p]} + \frac{(-1)^{p(n-p)}}{(n-p)!} \eta_{\rho_1 \dots \rho_{n-p} \mu_1 \dots \mu_p} u^{\rho_1} (\frac{\Sigma}{\vec{u}}H)^{\rho_2 \dots \rho_{n-p}} \\ \Sigma_{\mu_{p+1} \mu_{p+2} \dots \mu_n}^* &= -(n-p) u_{[\mu_{p+1}} (\frac{\Sigma}{\vec{u}}H)_{\mu_{p+2} \dots \mu_n]} - \frac{1}{p!} \eta_{\mu_1 \dots \mu_n} u^{\mu_1} (\frac{\Sigma}{\vec{u}}E)^{\mu_2 \dots \mu_p} \end{aligned}$$

which can be rewritten in a compact form as

$$\Sigma = -\mathbf{u} \wedge (\frac{\Sigma}{\vec{u}}\mathbf{E}) + (-1)^{p(n-p)} [\mathbf{u} \wedge (\frac{\Sigma}{\vec{u}}\mathbf{H})], \quad \Sigma^* = -\mathbf{u} \wedge (\frac{\Sigma}{\vec{u}}\mathbf{H}) - [\mathbf{u} \wedge (\frac{\Sigma}{\vec{u}}\mathbf{E})]$$

where \wedge denotes the exterior product. These formulae show that the E-H parts of Σ determine Σ , and that Σ vanishes iff its E-H parts are zero with respect to some (and then to all) \vec{u} . The general E-H decomposition for arbitrary r -fold forms does all this on each $[n_T]$ -block.

Obviously, one can put forward the following definition: a single p -form (or alternatively a single $[p]$ block of an r -fold form)

$$\Sigma_{[p]} \text{ is called } \begin{cases} \text{purely electric if } \exists \vec{u} \text{ such that } (\frac{\Sigma}{\vec{u}}\mathbf{H}) = \mathbf{0} \\ \text{purely magnetic if } \exists \vec{u} \text{ such that } (\frac{\Sigma}{\vec{u}}\mathbf{E}) = \mathbf{0} \\ \text{null if } \Sigma_{\rho\mu_2 \dots \mu_p} \Sigma^*_{\nu_2 \dots \nu_{n-p}} = 0 \text{ and } \Sigma_{\mu_1 \dots \mu_p} \Sigma^{\mu_1 \dots \mu_p} = 0 \end{cases}$$

2.2 Illustrative examples

As initial example, take a single 1-form A_μ with components $(A_0, A_1, \dots, A_{n-1})$ in any orthonormal (ON) basis $\{\vec{e}_\mu\}$. Its electric part with respect to $\vec{u} = \vec{e}_0$ is simply A_0 (that is, the time component), and the corresponding magnetic part is $(0, A_1, \dots, A_{n-1})$ (the spatial part). As is obvious, \mathbf{A} is purely electric, purely magnetic or null according to whether the vector \vec{A} is timelike, spacelike

or null, respectively. If we now consider a 2-form $F_{[2]}$ in the given ON basis

$$(F_{\mu\nu}) = \begin{pmatrix} 0 & \underline{F_{01}} & \underline{F_{02}} & \dots & \underline{F_{0\,n-1}} \\ & 0 & \underline{F_{12}} & \dots & \underline{F_{1\,n-1}} \\ & & 0 & \dots & \underline{F_{2\,n-1}} \\ & & & \dots & \dots \\ & & & & 0 \end{pmatrix}$$

its electric part is the spatial 1-form constituted by the underlined components while the rest of components provide the magnetic part, which is a spatial $(n-3)$ -form.

Going to the case of 2-fold (or double) forms, and starting with the simplest case of a double (1,1)-form $A_{\mu\nu}$ ($A_{(\mu\nu)} \neq 0$), the four E-H parts can be identified by writing $A_{[1][1]}$ in the ON basis above

$$(A_{\mu\nu}) = \begin{pmatrix} \boxed{A_{00}} & \underline{A_{01}} & \underline{A_{02}} & \dots & \underline{A_{0\,n-1}} \\ \underbrace{A_{10}} & A_{11} & A_{12} & \dots & A_{1\,n-1} \\ \underbrace{A_{20}} & A_{21} & A_{22} & \dots & A_{2\,n-1} \\ \dots & \dots & \dots & \dots & \dots \\ \underbrace{A_{n-1\,0}} & A_{n-1\,1} & A_{n-1\,2} & \dots & A_{n-1\,n-1} \end{pmatrix}$$

so that the ‘EE’ part is given by the scalar A_{00} in the box, the ‘EH’ part is given by the underlined components, the ‘HE’ part is constituted by the underbraced components, and the ‘HH’ part is the double $(n-2, n-2)$ -form given by the rest. In the case that $A_{\mu\nu} = A_{(\mu\nu)}$ is symmetric, the EH and HE parts are equal and the HH part is symmetric too. Some particular cases are of interest here. For instance, the metric itself is a double (1,1)-form and admits its own E-H decomposition. Its EE part is the ‘ g_{00} ’ component (related to the Newtonian potential), and its EH=HE part, corresponding to the ‘ g_{0i} ’ in general, can always be set to zero by choice of orthonormal basis, and as is known, these ON basis can be further chosen to be *locally* holonomous (Gaussian coordiantes). At this level, this means that the gravimagnetic effects are always *locally* inertial, that is, due to the reference frame. Another case of interest is the general energy-momentum tensor $T_{\mu\nu}$. Comparing with its canonical decomposition along \vec{u} (see e.g. ¹²), its EE part corresponds to the energy density, its EH=HE part is the energy-flux vector, and the HH part is given by the pressures and/or tensions, all of these with regard to \vec{u} .

Let us consider finally the case of double (2,2)-forms, which in particular contain the curvature tensors. In general, such a $K_{[2][2]}$ has $n^2(n-1)^2/4$ independent components, and they are distributed as follows: its EE part is

a double (1,1)-form carrying $(n-1)^2$ independent components, its EH and HE parts are double $(1, n-3)$ - and $(n-3, 1)$ -forms respectively and have $(n-1)^2(n-2)/2$ components each, and its HH part is a double $(n-3, n-3)$ -form with the remaining $(n-1)^2(n-2)^2/4$ components. However, some particular cases arise for the curvature tensors because, for instance, the Riemann tensor satisfies the first Bianchi identity $R_{\alpha[\mu\nu\rho]} = 0$. This implies that $R_{[2][2]}$ has in fact $n^2(n^2-1)/12$ independent components, which are distributed into $n(n-1)/2$ components for the EE part, $n(n-1)(n-2)/3$ for the EH (which is in this case equivalent to the HE) part, and $n(n-1)^2(n-2)/12$ for the HH part. Similarly, the Weyl tensor has these symmetry properties and is also traceless. This implies that the $n(n+1)(n+2)(n-3)/12$ independent components of $C_{[2][2]}$ reorganize into $(n+1)(n-2)/2$ for its EE part, $(n^2-1)(n-3)/3$ for its EH (or HE) part and $n(n^2-1)(n-4)/12$ for its HH part. Notice that, for $n=4$, the HH part of the Weyl tensor has no *new* independent components, and this is why in 4 dimensions only one electric (corresponding to EE=HH) and one magnetic part (EH=HE) arise, compare with subsection 1.2.

3 Super-energy tensors

As an application of the above, for any $t_{\mu_1 \dots \mu_m}$ and any \vec{u} one can define

$$W_t(\vec{u}) \equiv \frac{1}{2} \left([\vec{u} \underbrace{EE \dots E}_r]^2 + \dots + [\vec{u} \underbrace{HH \dots H}_r]^2 \right) \geq 0$$

where $[\]^2$ means contraction of all indices (divided by the appropriate factors⁵). $W_t(\vec{u})$ is obviously a non-negative quantity and it vanishes iff $t_{\mu_1 \dots \mu_m}$ is zero. In fact, in any ON basis one can write

$$W_t(\vec{e}_0) = \frac{1}{2} \sum_{\mu_1, \dots, \mu_m=0}^{n-1} |t_{\mu_1 \dots \mu_m}|^2.$$

Comparing with (3) we see that $W_t(\vec{u})$ has the *mathematical* properties of an energy density; being not an energy in general, though, it is usually called^{10,11,3,5} the “super-energy” (s-e) density of the field $t_{\mu_1 \dots \mu_m}$ with regard to \vec{u} . The natural question arises if one can define an appropriate s-e analogue of the Maxwell tensor (1). The answer is positive and in fact dates back to the gravitational s-e tensors found by Bel^{4,10,11} and independently by Robinson. The universal generalization of the s-e tensors has recently been achieved^{3,5} as follows. First of all, the notation $t^{\mathcal{P}}$ with $\mathcal{P} = 1, \dots, 2^r$ is used to denote all duals of t , where $\mathcal{P} = 1 + \sum_{\Upsilon=1}^r 2^{\Upsilon-1} \epsilon_{\Upsilon}$ and ϵ_{Υ} is one or zero according to whether the block

$[n_\Upsilon]$ is dualized or not. One can define a product \odot of an r -fold form by itself resulting in a $2r$ -tensor

$$(t \odot t)_{\lambda_1 \mu_1 \dots \lambda_r \mu_r} \equiv \left(\prod_{\Upsilon=1}^r \frac{1}{(n_\Upsilon - 1)!} \right) \tilde{t}_{\lambda_1 \rho_2 \dots \rho_{n_1}, \dots, \lambda_r \sigma_2 \dots \sigma_{n_r}} \tilde{t}_{\mu_1}^{\rho_2 \dots \rho_{n_1}, \dots, \sigma_2 \dots \sigma_{n_r}}.$$

From each block in $t_{[n_1] \dots [n_r]}$ two indices are obtained in $(t \odot t)_{\lambda_1 \mu_1 \dots \lambda_r \mu_r}$. With this at hand, the general definition of the basic s-e tensor of t is^{3,5}:

$$T_{\lambda_1 \mu_1 \dots \lambda_r \mu_r} \{t\} \equiv \frac{1}{2} \sum_{\mathcal{P}=1}^{2^r} (t^{\mathcal{P}} \odot t^{\mathcal{P}})_{\lambda_1 \mu_1 \dots \lambda_r \mu_r}. \quad (8)$$

Observe that any dual $t^{\mathcal{P}}$ of the original tensor $t(= t^1)$ generates the same basic s-e tensor (8). Therefore, one only needs to consider blocks with at most $n/2$ indices if n is even, or $(n-1)/2$ if n is odd. We also remark that $t_{[n_1] \dots [n_r]}$ could contain n -blocks (with dual 0-blocks) for which the expression (8) has no meaning. However, an n -form in n dimensions is trivial and contributes to the superenergy with a scalar factor given precisely by its dual 0-form squared. We note that $T_{\lambda_1 \mu_1 \dots \lambda_r \mu_r} \{t\} = T_{(\lambda_1 \mu_1) \dots (\lambda_r \mu_r)} \{t\}$ and if the tensor $t_{[n_1] \dots [n_r]}$ is symmetric in the interchange $[n_\Upsilon] \leftrightarrow [n_{\Upsilon'}]$ ($n_\Upsilon = n_{\Upsilon'}$), then the s-e tensor (8) is symmetric in the interchange of the corresponding $(\lambda_\Upsilon \mu_\Upsilon)$ - and $(\lambda_{\Upsilon'} \mu_{\Upsilon'})$ -pairs. Also, if n is even, then (8) is traceless in any $(\lambda_\Upsilon \mu_\Upsilon)$ -pair with $n_\Upsilon = n/2$. It is remarkable that, after expanding all duals in (8), one obtains an explicit expression for the s-e tensor which is *independent* of the dimension n , see⁵. Finally, one can prove the important result that if $t_{[n_1] \dots [n_r]}$ is decomposable as in (7), then

$$T_{\lambda_1 \mu_1 \dots \lambda_r \mu_r} \{t_{[n_1] \dots [n_r]}\} = T_{\lambda_1 \mu_1} \left\{ \Omega_{[n_1]}^{(1)} \right\} \dots T_{\lambda_r \mu_r} \left\{ \Omega_{[n_r]}^{(r)} \right\}. \quad (9)$$

The timelike component of $T_{\lambda_1 \mu_1 \dots \lambda_r \mu_r} \{t\}$ with respect to an observer \vec{u} is precisely the s-e density $W_t(\vec{u})$ defined above. More importantly, the s-e tensors have the fundamental property that they always satisfy a generalization of the dominant energy condition (4) called the *dominant property*, see section 4. The s-e tensor $T_{\lambda_1 \mu_1 \dots \lambda_r \mu_r} \{t\}$ and its derived tensors by permutation of indices are the *only* (up to linear combinations) tensors quadratic in t and with the dominant property⁵. Therefore, $T_{(\lambda_1 \mu_1) \dots (\lambda_r \mu_r)} \{t\}$ is, up to a trivial factor, the unique *completely symmetric* tensor quadratic in t with the dominant property. This property of general s-e tensors was used in¹⁵ to find criteria for the causal propagation of fields on n -dimensional spacetimes.

In 4 dimensions, the s-e tensor of a 2-form $F_{ab} = F_{[ab]}$ is its Maxwell energy-momentum tensor (1), and the s-e tensor of an exact 1-form $d\phi$ has the form (in any n after expanding duals)

$$T_{\mu\nu}\{\nabla_{[1]}\phi\} = \nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}(\nabla_\rho\phi\nabla^\rho\phi)g_{\mu\nu} \quad (10)$$

which is exactly the energy-momentum tensor for a massless scalar field ϕ . If we compute the s-e tensor of the Riemann tensor we get the so-called Bel tensor (see ^{16,10} for $n = 4$)

$$\begin{aligned} 2T_{\alpha}{}^{\beta}{}_{\lambda}{}^{\mu}\{R_{[2],[2]}\} &= R_{\alpha\rho,\lambda\sigma}R^{\beta\rho,\mu\sigma} + \frac{1}{(n-3)!}R_{\alpha\rho,\lambda\sigma_4\dots\sigma_n}{}^*R^{\beta\rho,\mu\sigma_4\dots\sigma_n}{}^* + \\ &\frac{1}{(n-3)!}R_{\alpha\rho_4\dots\rho_n,\lambda\sigma}{}^*R^{\beta\rho_4\dots\rho_n,\mu\sigma} + \frac{1}{[(n-3)!]^2}R_{\alpha\rho_4\dots\rho_n,\lambda\sigma_4\dots\sigma_n}{}^*R^{\beta\rho_4\dots\rho_n,\mu\sigma_4\dots\sigma_n}{}^* \end{aligned}$$

which, after expanding the duals, becomes independent of n :

$$\begin{aligned} B_{\alpha\beta\lambda\mu} \equiv T_{\alpha\beta\lambda\mu}\{R_{[2],[2]}\} &= R_{\alpha\rho,\lambda\sigma}R_{\beta}{}^{\rho}{}_{\mu}{}^{\sigma} + R_{\alpha\rho,\mu\sigma}R_{\beta}{}^{\rho}{}_{\lambda}{}^{\sigma} - \\ &-\frac{1}{2}g_{\alpha\beta}R_{\rho\tau,\lambda\sigma}R^{\rho\tau}{}_{\mu}{}^{\sigma} - \frac{1}{2}g_{\lambda\mu}R_{\alpha\rho,\sigma\tau}R_{\beta}{}^{\rho,\sigma\tau} + \frac{1}{8}g_{\alpha\beta}g_{\lambda\mu}R_{\rho\tau,\sigma\nu}R^{\rho\tau,\sigma\nu}. \end{aligned} \quad (11)$$

The s-e tensor of the Weyl curvature has exactly the same expression as (11) by replacing $C_{\alpha\rho,\lambda\sigma}$ for $R_{\alpha\rho,\lambda\sigma}$, and is called the generalized Bel-Robinson tensor ^{14,11,10,5}. In 4 and 5 dimensions it is completely symmetric, and also traceless for $n = 4$, but not in general. Of course, the Bel and Bel-Robinson tensors coincide in Ricci-flat spacetimes. The dominant property of the Bel-Robinson tensor was used by Christodoulou and Klainerman ¹⁷ in their study of the global stability of Minkowski spacetime, and in ¹⁸ to prove the causal propagation of gravity in vacuum for $n = 4$.

By using the second Bianchi identity $\nabla_{[\nu}R_{\alpha\beta]\lambda\mu} = 0$, it follows from (11)

$$\nabla_\alpha B^{\alpha\beta\lambda\mu} = R_{\rho}{}^{\lambda}{}_{\sigma}j^{\mu\sigma\rho} + R_{\rho}{}^{\mu}{}_{\sigma}j^{\lambda\sigma\rho} - \frac{1}{2}g^{\lambda\mu}R_{\rho\sigma\gamma}{}^{\beta}j^{\sigma\gamma\rho} \quad (12)$$

where $j_{\lambda\mu\beta} \equiv \nabla_\lambda R_{\mu\beta} - \nabla_\mu R_{\lambda\beta}$. This is analogous to the formula shown for $\tau_{\mu\nu}$ in subsection 1.1 and, again, it leads to the interesting result

$$j_{\lambda\mu\beta} = 0 \quad \implies \quad \nabla_\rho B^{\rho\beta\lambda\mu} = 0.$$

In particular, the Bel (and Bel-Robinson) tensor is divergence-free in Ricci-flat or Einstein spacetimes. If there is a Killing vector $\tilde{\xi}$ we get

$$j_{\lambda\mu\beta} = 0 \quad \implies \quad \nabla_\rho J^\rho = 0, \quad J^\rho(R_{[2][2]}; \tilde{\xi}) \equiv B^{\rho\beta\lambda\mu}\tilde{\xi}_\beta\tilde{\xi}_\lambda\tilde{\xi}_\mu \quad (13)$$

so that again the *divergence-free current* \vec{J} can be used to construct conserved quantities with the Gauss theorem. When the charge current \vec{j} for the electromagnetic field, or the $j_{\lambda\mu\beta}$ for the gravitational field, are not zero the corresponding currents are no longer divergence-free. In the case of classical Electromagnetism this was interpreted as a signal of the existence of energy-momentum associated to the charges creating the field and appearing in \vec{j} , see e.g.¹⁹. When one constructs the energy-momentum tensor corresponding to these charges and uses the field equations, the *total* energy-momentum tensor is again divergence-free⁹ and can be used to construct the conserved *mixed* quantities. Can one find similar results for the case of the gravitational field and the Bel tensor?

A positive answer to this question requires, at least, the definition of s-e tensors of the Bel type for physical fields. However, this can be achieved by using our general definition (8) and following an idea launched long ago by Chevreton²⁰ and recently retaken by other authors^{21,22} in the case of Minkowski spacetime. The combined idea was put forward in^{3,4,5} and simply uses the covariant derivatives of the physical fields to construct the appropriate s-e tensors. As an outstanding example, we can define the basic s-e tensor of the electromagnetic field as that corresponding to the double (1,2)-form $\nabla_\alpha F_{\mu\nu}$

$$E_{\alpha\beta\lambda\mu} \equiv T_{\alpha\beta\lambda\mu}\{\nabla_{[1]}F_{[2]}\} = \nabla_\alpha F_{\lambda\rho}\nabla_\beta F_\mu{}^\rho + \nabla_\alpha F_{\mu\rho}\nabla_\beta F_\lambda{}^\rho - \\ g_{\alpha\beta}\nabla_\sigma F_{\lambda\rho}\nabla^\sigma F_\mu{}^\rho - \frac{1}{2}g_{\lambda\mu}\nabla_\alpha F_{\sigma\rho}\nabla_\beta F^{\sigma\rho} + \frac{1}{4}g_{\alpha\beta}g_{\lambda\mu}\nabla_\tau F_{\sigma\rho}\nabla^\tau F^{\sigma\rho}$$

whose symmetry properties are $E_{\alpha\beta\lambda\mu} = E_{(\alpha\beta)(\lambda\mu)}$ and has zero divergence in flat spacetimes⁵. The tensor $E_{(\alpha\beta\lambda\mu)}$ is unique in the sense commented on above and coincides with the symmetric part of Chevreton's²⁰, and can be used to construct conserved quantities along shock-waves with continuous $F_{[2]}$, see^{23,3,5}. This is a first indication of the possibility of the interchange of 'superenergy' quantities between different fields. However, the complete analysis in the Einstein-Maxwell case is still to be done.

Another exact and interesting result was achieved, though, in the simpler case of a scalar field ϕ coupled to gravity. In order to get the basic s-e tensor of a *massless* scalar field one can use the double symmetric (1,1)-form $\nabla_\alpha\nabla_\beta\phi$ as starting object, so that the corresponding tensor (8) becomes

$$S_{\alpha\beta\lambda\mu} \equiv T_{\alpha\beta\lambda\mu}\{\nabla_{[1]}\nabla_{[1]}\phi\} = \nabla_\alpha\nabla_\lambda\phi\nabla_\mu\nabla_\beta\phi + \nabla_\alpha\nabla_\mu\phi\nabla_\lambda\nabla_\beta\phi - \\ -g_{\alpha\beta}\nabla_\lambda\nabla^\rho\phi\nabla_\mu\nabla_\rho\phi - g_{\lambda\mu}\nabla_\alpha\nabla^\rho\phi\nabla_\beta\nabla_\rho\phi + \frac{1}{2}g_{\alpha\beta}g_{\lambda\mu}\nabla_\sigma\nabla_\rho\phi\nabla^\sigma\nabla^\rho\phi \quad (14)$$

In fact this tensor was previously found by Bel²¹ and Teyssandier²² in Special Relativity ($n = 4$). Its symmetry properties are $S_{\alpha\beta\lambda\mu} = S_{(\alpha\beta)(\lambda\mu)} =$

$S_{\lambda\mu\alpha\beta}$. Concerning its divergence, a straightforward computation using the Ricci identity⁷ leads to

$$\begin{aligned}\nabla_\alpha S^\alpha{}_{\beta\lambda\mu} = & 2\nabla_\beta \nabla_{(\lambda} \phi R_{\mu)\rho} \nabla^\rho \phi - g_{\lambda\mu} R^{\sigma\rho} \nabla_\beta \nabla_\rho \phi \nabla_\sigma \phi - \\ & - \nabla_\sigma \phi \left(2\nabla^\rho \nabla_{(\lambda} \phi R_{\mu)\rho\beta} + g_{\lambda\mu} R_{\rho\beta\tau} \nabla^\rho \nabla^\tau \phi \right)\end{aligned}$$

so that all the terms on the righthand side involve components of the Riemann tensor. Thus, in any flat region of the spacetime (that is, with vanishing Riemann tensor, so that there is no gravitational field), the s-e tensor of the massless scalar field (14) is divergence-free. This leads to conserved currents of type (13) for the scalar field in flat spacetimes^{4,5,22}. For a deeper study of these in Special Relativity see²².

The situation is therefore that the Bel tensor is divergence-free in Ricci-flat spacetimes, and the s-e tensor (14) is divergence-free in the absence of gravitational field. These divergence-free properties lead to conserved currents of type (13) whenever there are symmetries in the spacetime, see⁵ for a lengthy discussion. Thus, the important question arises if one can combine the Bel tensor with the s-e tensor (14) to produce a conserved current in the *mixed* case when there are both a scalar field and the curvature that it generates. Assume therefore that the Einstein-Klein-Gordon equations for a minimally coupled massless scalar field hold^b, and that there is a Killing vector ξ in the spacetime. A simple calculation^{4,5} leads to

$$\nabla_\alpha J^\alpha = 0, \quad \text{where} \quad J^\alpha \left(R_{[2][2]}, \nabla_{[1]} \nabla_{[1]} \phi; \xi \right) \equiv \left(B^{\alpha\beta\lambda\mu} + S^{\alpha\beta\lambda\mu} \right) \xi_\beta \xi_\lambda \xi_\mu.$$

Notice that only the completely symmetric parts of B and S are relevant here. This provides conserved quantities proving the exchange of s-e properties between the gravitational and scalar fields, because neither $B^{\alpha\beta\lambda\mu} \xi_\beta \xi_\lambda \xi_\mu$ nor $S^{\alpha\beta\lambda\mu} \xi_\beta \xi_\lambda \xi_\mu$ are divergence-free separately (if the other field is present!).

4 Causal tensors

As mentioned in Section 3, the s-e tensors satisfy a generalization of the dominant energy condition (4) given in the following

Definition 4.1 *A tensor $T_{\mu_1 \dots \mu_s}$ is said to have the dominant property if*

$$T_{\mu_1 \dots \mu_s} v_1^{\mu_1} \dots v_s^{\mu_s} \geq 0$$

for any set $\vec{v}_1, \dots, \vec{v}_s$ of causal future-pointing vectors. The set of tensors with the dominant property is denoted by \mathcal{DP} , and $-\mathcal{DP}$ will mean the set of tensors $T_{\mu_1 \dots \mu_s}$ such that $-T_{\mu_1 \dots \mu_s} \in \mathcal{DP}$.

^bActually, the result holds true for massive fields too.^{4,5}

Notice that $\mathcal{DP} \cap -\mathcal{DP} = \{0\}$. The non-negative real numbers are considered a special case with the dominant property, $\mathbb{R}^+ \subset \mathcal{DP}$. Rank-1 tensors with the dominant property are the past-pointing causal vectors, while those in $-\mathcal{DP}$ are the future-directed ones. For rank-2 tensors, the dominant property is exactly (4), usually called the dominant energy *condition* for it is a requirement placed on physically acceptable energy-momentum tensors^{6,7}. As in the case of past- and future-pointing vectors, any statement concerning \mathcal{DP} has its counterpart concerning $-\mathcal{DP}$, and they will be omitted sometimes. The elements of $\mathcal{DP} \cup -\mathcal{DP}$ will thus be called “causal tensors”, and this set has the mathematical structure of a *cone* because

$$T_{\mu_1 \dots \mu_s}, S_{\mu_1 \dots \mu_s} \in \mathcal{DP} \implies a T_{\mu_1 \dots \mu_s} + b S_{\mu_1 \dots \mu_s} \in \mathcal{DP} \quad \forall a, b \in \mathbb{R}^+.$$

Therefore, this seems to be an adequate generalization of the solid Lorentz cone to tensors of arbitrary rank. A natural question that arises is: what are the boundaries of these solid cones, or in other words, which are the higher-rank generalizations of the null cone? Actually, we also have $\forall T_1, T_2 \in \mathcal{DP}$, $T_1 \otimes T_2 \in \mathcal{DP}$ ²⁴, so that $\mathcal{DP} \cup -\mathcal{DP}$ is a sort of graded algebra of solid cones.

The main properties of \mathcal{DP} are summarized in what follows, see also²⁴ and Bergqvist’s contribution to this volume. To that end, the following notations will be used: a) for any two tensors $T_{\mu_1 \dots \mu_r}^{(1)}$ and $T_{\mu_1 \dots \mu_s}^{(2)}$, we write

$$(T^{(1)} \times_j T^{(2)})_{\mu_1 \dots \mu_{r+s-2}} \equiv T_{\mu_1 \dots \mu_{i-1} \rho \mu_i \dots \mu_{r-1}}^{(1)} T_{\mu_r \dots \mu_{r+j-2} \rho \mu_{r+j-1} \dots \mu_{r+s-2}}^{(2)}$$

so that a contraction with the i^{th} index of the first tensor and the j^{th} of the second is taken. b) In particular, for any two rank-2 tensors $A_{\mu\nu}$ and $B_{\mu\nu}$ we will simply write $A \times B \equiv A_1 \times_1 B$, which is another rank-2 tensor. c) For any two tensors $T_{\mu_1 \dots \mu_s}, S_{\mu_1 \dots \mu_s}$ of the same rank we put $T \cdot S \equiv T_{\mu_1 \dots \mu_s} S^{\mu_1 \dots \mu_s}$.

Several characterizations of the set \mathcal{DP} are the following

$$\begin{aligned} T_{\mu_1 \dots \mu_s} \in \mathcal{DP} &\iff T_{\mu_1 \dots \mu_s} u_1^{\mu_1} \dots u_s^{\mu_s} > 0, \quad \forall \text{timelike } \vec{u}_1, \dots, \vec{u}_s \iff \\ &\iff T_{\mu_1 \dots \mu_s} k_1^{\mu_1} \dots k_s^{\mu_s} \geq 0, \quad \forall \text{null } \vec{k}_1, \dots, \vec{k}_s \iff \\ &\iff T_i \times_j S \in \mathcal{DP}, \quad \forall S \in -\mathcal{DP} \iff T_i \times_j t \in -\mathcal{DP}, \quad \forall t \in \mathcal{DP} \\ &\iff T_{0 \dots 0} \geq |T_{\alpha_1 \dots \alpha_s}| \text{ in any ON basis } \{\vec{e}_\mu\} \text{ with a future } \vec{e}_0, \end{aligned}$$

while some characterizations of causal tensors are

$$\begin{aligned} 0 \neq T \in \mathcal{DP} \cup -\mathcal{DP} &\iff 0 \neq T_i \times_i T \in -\mathcal{DP} \text{ for some } i \\ &\iff 0 \neq T_i \times_j T \in -\mathcal{DP} \quad \forall i, j \end{aligned}$$

or the extreme cases given by

$$\begin{aligned} T_i \times_i T &= 0 \quad \forall i \quad \Longleftrightarrow \\ \Longleftrightarrow T &= \mathbf{k}_1 \otimes \dots \otimes \mathbf{k}_s \text{ for a set of } \underline{\text{null}} \mathbf{k}_1, \dots, \mathbf{k}_s \implies T \in \mathcal{DP} \cup -\mathcal{DP}. \end{aligned}$$

If $T_{\mu_1 \dots \mu_s}$ is antisymmetric in any two indices then $T_{\mu_1 \dots \mu_s} \notin \mathcal{DP} \cup -\mathcal{DP}$. In particular, p -forms Σ with $p > 1$ cannot be in $\mathcal{DP} \cup -\mathcal{DP}$. Nevertheless, we will still use the name *causal* p -form for the single forms Σ such that $\Sigma \cdot \Sigma \leq 0$. Recall that a p -form $\Omega_{\nu_1 \dots \nu_p} = \Omega_{[\nu_1 \dots \nu_p]}$ is called simple if it is a product of p linearly independent 1-forms $\omega^1, \dots, \omega^p$, i.e. $\Omega = \omega^1 \wedge \dots \wedge \omega^p$. A p -form $\Omega_{[p]}$ is simple iff $\Omega_{[N-p]}^*$ is simple, and if and only if $\Omega_{[p]} \times_1 \Omega_{[N-p]}^* = 0$, see e.g. 25,26. According to the definition given in subsection 2.1, a p -form Ω is null if it is simple and causal with $\Omega \cdot \Omega = 0$.

Let us introduce the following classes: \mathcal{SE} will denote the set of all s-e tensors (8) introduced in section 3 plus the rank-2 tensors of type $fg_{\mu\nu}$ with $f \leq 0$; by \mathcal{SS} is meant the set of s-e tensors of *simple* p -forms plus the tensors of type $fg_{\mu\nu}$ with $f \leq 0$; finally \mathcal{NS} denotes the set of s-e tensors of *null* p -forms. By $-\mathcal{SE}$, $-\mathcal{SS}$ and $-\mathcal{NS}$ we mean the sets of tensors T such that $-T \in \mathcal{SE}$, \mathcal{SS} or \mathcal{NS} , respectively. Notice that all tensors in \mathcal{NS} and \mathcal{SS} are of rank 2, and that $\mathcal{NS} \subset \mathcal{SS} \subset \mathcal{SE}$. Furthermore, $\mathcal{SE} \subset \mathcal{DP}$, as was proved in²⁷ for $n = 4$ using spinors and later for general n in^{5,28}. An important result of relevance in General relativity is²⁴

Proposition 4.1 *In dimension $n \leq 4$, $T_{\mu\nu} \in \mathcal{SE} \implies (T \times T)_{\mu\nu} = h^2 g_{\mu\nu}$.*

This result does *not* hold for $n > 4$, and one wonders what is its adequate generalization to any n . Similarly, it is desirable to know the converse of Proposition 4.1, as this is the basis of the famous Rainich conditions^{29,30,31} which allow to prove the existence of an electromagnetic field based only on the underlying geometry if $T_\mu{}^\mu = 0$, see (2). These questions are to be answered now and in the next section, and are actually related to that asked before about the ‘higher-rank’ generalizations of the null cone.

To begin with, we have²⁴

Proposition 4.2 $T_{\mu\nu} \in \mathcal{SS} \implies (T \times T)_{\mu\nu} = h^2 g_{\mu\nu}$.

This result holds in arbitrary n and generalizes Proposition 4.1. Furthermore, we also have the following fundamental result

Theorem 4.1 *In n dimensions, any symmetric rank-2 tensor $S_{(\mu\nu)} = S_{\mu\nu} \in \mathcal{DP}$ can be written*

$$S_{\mu\nu} = T_{\mu\nu}\{\Omega_{[1]}\} + \dots + T_{\mu\nu}\{\Omega_{[n-1]}\} - \alpha g_{\mu\nu} \quad (15)$$

where $\alpha \geq 0$ and $T_{\mu\nu}\{\Omega_{[p]}\} \in \mathcal{SS}$ are the *superenergy tensors of simple and causal* p -forms $\Omega_{[p]}$, $p = 1, \dots, n-1$ such that for $p = 2, \dots, n-1$ they have

the structure $\Omega_{[p]} = \mathbf{k}^1 \wedge \dots \wedge \mathbf{k}^p$ where $\mathbf{k}^1, \dots, \mathbf{k}^p$ are appropriate null 1-forms. The number of tensors on the righthand side of (15) and the structure of the $\Omega_{[p]}$ depend on the particular $S_{\mu\nu}$ as follows: if $S_{\mu\nu}$ has $n - q \geq 1$ null eigenvectors $\vec{k}^1, \dots, \vec{k}^{n-q}$ then at least $T_{\mu\nu}\{\Omega_{[n-q]}\}$, with $\Omega_{[n-q]} = \mathbf{k}^1 \wedge \dots \wedge \mathbf{k}^{n-q}$, must appear in (15), and possibly those to its right in (15). If it has no null eigenvectors, then at least $T_{\mu\nu}\{\Omega_{[1]}\}$ appears in (15), and possibly those to its right, and $\Omega_{[1]}$ is timelike.

Remark: As already mentioned, the s-e tensor of the dual of a p -form is identical with that of the p -form itself. Thus, in the sum (15) there are two s-e tensors of 1-forms, namely $T_{\mu\nu}\{\Omega_{[1]}\}$ and $T_{\mu\nu}\{\Omega_{[1]}^*\} = T_{\mu\nu}\{\Omega_{[n-1]}\}$, but the first one corresponds to a *causal* 1-form and the second to a *spacelike* one. Similar remarks apply to the 2-forms $\Omega_{[2]}$ and $\Omega_{[n-2]}^*$, and so on.

Theorem 4.1 means that the elements in \mathcal{SS} can be used to build up \mathcal{DP} and, in this sense, \mathcal{SS} seems a generalization of the null cone to rank-2 tensors. One can compare Theorem 4.1 with the standard result that any causal and past-directed vector \vec{v} can be expressed as the combination of two past-pointing null vectors \vec{k}^1, \vec{k}^2 : $\vec{v} = \vec{k}^1 + \alpha \vec{k}^2$ with $\alpha \geq 0$. Recall that this can be achieved in infinitely many ways. In the same manner, a symmetric $S_{\mu\nu} \in \mathcal{DP}$ can be expressed as a sum of n elements of \mathcal{SS} in many ways. In the above Theorem 4.1, however, the representation of $S_{\mu\nu} \in \mathcal{DP}$ has been chosen in a canonical way in relation with the null eigenvectors of $S_{\mu\nu}$.

Theorem 4.1 allows also to prove the converse of Proposition 4.2, see²⁴, in the following sense

Theorem 4.2 *If $T_{\mu\nu}$ is symmetric and $(T \times T)_{\mu\nu} = f g_{\mu\nu}$ then:*

- (a) $f = 0 \implies T_{\mu\nu} \in \mathcal{NS} \cup -\mathcal{NS}$ and $T_{\mu\nu} = k_\mu k_\nu$ for a null \mathbf{k} .
- (b) $f \neq 0 \implies T_{\mu\nu} \in \mathcal{SS} \cup -\mathcal{SS}$. Moreover, $\epsilon T_\mu{}^\mu / \sqrt{T_{\rho\sigma} T^{\rho\sigma} / n}$ takes one of the values $\{n, (n-2), \dots, (2p-n), \dots, (2-n)\}$ according to the rank p of the simple p -form $\Omega_{[p]}$ generating $T_{\mu\nu} \in \mathcal{SS} \cup -\mathcal{SS}$, where $\Omega_{[p]}$ is causal of the type used in Theorem 4.1 and $\epsilon^2 = 1$.

Proposition 4.1 and Theorem 4.2 can be combined to give

Corollary 4.1 *If $n \leq 4$ then $\mathcal{SE} = \mathcal{SS}$.*

For $n = 4$ this means that the energy-momentum tensor (1) of any Maxwell field $F_{[2]}$, be it simple or not, coincides with the energy-momentum of (possibly) another simple 2-form. This is well known, related to the duality rotations^{26,32}, and forms the basis of the algebraic Rainich conditions^{29,26,30,31}, statable as

Corollary 4.2 (Algebraic classical Rainich's conditions) *In $n = 4$, a tensor $T_{\mu\nu}$ is (up to sign) algebraically the energy-momentum tensor of a 2-form, that is, it takes the form (1), if and only if $(T \times T)_{\mu\nu} = f g_{\mu\nu}$, $T_{\mu\nu} = T_{(\mu\nu)}$ and $T_\mu{}^\mu = 0$, which are relations (2).*

All the above can also be seen from the point of view of Lorentz transformations and some of its generalizations. To that end, let us introduce the following terminology: $T_\mu{}^\nu$ is said to be a *null-cone preserving map* if $k^\mu T_\mu{}^\nu$ is null for any null vector \vec{k} . A map that preserves the null cone is said to be *orthochronous* (respectively *time reversal*) if it keeps (resp. reverses) the cone's time orientation, and is called *proper*, *improper* or *singular* if $\det(T_\mu{}^\nu)$ is positive, negative, or zero, respectively. If the map is proper and orthochronous then it is called *restricted*. A null-cone preserving map is *involutory* if $T_\mu{}^\nu = (T^{-1})_\mu{}^\nu$, and *bi-preserving* if $T_\mu{}^\nu k_\nu$ is also null for any null 1-form \vec{k} . Notice that involutory null-cone preserving maps are necessarily non-singular. The following lemma gives a geometrical interpretation to some previous results.

Lemma 4.1 $(T_2 \times_2 T)_{\mu\nu} = f g_{\mu\nu} \iff T_\mu{}^\nu$ is a null cone preserving map.

This allows to prove²⁴ that, in fact, $(\mathcal{SS} \cup -\mathcal{SS}) \setminus (\mathcal{NS} \cup -\mathcal{NS})$ is the set of tensors proportional to involutory Lorentz transformations, while $\mathcal{NS} \cup -\mathcal{NS}$ is a subset of the *singular* null-cone bi-preserving maps. They can be easily classified into orthochronous or time-reversal, and proper or improper using the above results, see²⁴.

5 Generalization of algebraic Rainich conditions

Finally, let us see how can one use all the previous results to provide algebraic Rainich-like conditions in any dimension n . To start with, for a massless scalar field we have

Corollary 5.1 *In n dimensions, a tensor $T_{\mu\nu}$ is algebraically the energy-momentum tensor (10) of a minimally coupled massless scalar field ϕ if and only if $T_{\mu\nu} \in \mathcal{SS}$ and $T_\mu{}^\mu = \beta \sqrt{T_{\mu\nu} T^{\mu\nu}}/n$ where $\beta = \pm(n-2)$. Moreover, $d\phi$ is spacelike or timelike if $T_\mu{}^\mu \neq 0$ and $\beta = n-2$ or $\beta = 2-n$, respectively, and null if $T_\mu{}^\mu = 0 = T_{\mu\nu} T^{\mu\nu}$.*

This simple, complete and general result is to be compared with alternative approaches to the problem^{33,34,35,36} for $n = 4$. Of course, the Corollary follows almost immediately from Theorem 4.2.

We can also attack the problem of algebraic Rainich conditions for a perfect fluid in arbitrary dimension n .

Corollary 5.2 *A tensor $T_{\mu\nu}$ is algebraically the energy-momentum tensor of a perfect fluid satisfying the dominant energy condition if and only if*

$$T_{\mu\nu} = -\frac{\lambda}{2} g_{\mu\nu} + \mu T_{\mu\nu} \{v_{[1]}\} \quad (16)$$

where $\lambda, \mu \geq 0$ and \mathbf{v} is timelike, and therefore the tensor $T_{\mu\nu} \{v_{[1]}\} \in \mathcal{SS}$ is intrinsically characterized according to its trace, see Theorem 4.2. The velocity

vector of the fluid, its energy density and its pressure are given by $\vec{u} = -\vec{v}/(v \cdot v)$, $\rho = (-\mu(v \cdot v) + \lambda)/2$ and $P = -(\mu(v \cdot v) + \lambda)/2$, respectively.

The proof is quite simple again. Recall that a perfect fluid has the Segre type $\{1, (1 \dots 1)\}$, so that

$$T_{\mu\nu} = (\rho + P)u_\mu u_\nu + P g_{\mu\nu} \quad (17)$$

where $(u \cdot u) = -1$. Thus, if (16) holds it is obvious that $T_{\mu\nu}$ takes the form (17). Conversely, if (17) holds, then $T_{\mu\nu} - T_{\mu\nu}\{u_{[1]}\}$ has every null \vec{k} as eigenvector, as can be trivially checked. Therefore, $T_{\mu\nu} - T_{\mu\nu}\{u_{[1]}\}$ is proportional to the metric, as follows from Theorem 4.1 and the proportionality factor is obtained from the trace. In fact, we can rederive the conditions found for $n = 4$ in^{37,38} generalized to n dimensions as follows. From (16) we get

$$(T \times T)_{\mu\nu} = \lambda T_{\mu\nu} - \rho P g_{\mu\nu}$$

and also $n(T \times T)_\mu{}^\mu - (T_\mu{}^\mu)^2 \geq 0$, $T_\mu{}^\mu \geq n\lambda/2$ and $T_{\mu\nu}w^\mu w^\nu \geq \lambda/2$ for all timelike \vec{w} .

As a final example, let us consider the case of dust ($P = 0$ perfect fluids). Of course, the characterization of dust can be deduced from the previous one by setting $P = 0$. However, there are some particular cases in which some stronger results can be derived. For instance

Corollary 5.3 *In 5 dimensions, a tensor $T_{\mu\nu}$ is algebraically the energy-momentum tensor of a dust $T_{\mu\nu} = \rho u_\mu u_\nu$ where $(u \cdot u) = -1$ and $\rho \geq 0$ if and only if $T_{\mu\nu}$ is the s-e tensor of a 2-form $F_{[2]}$ with no null eigenvector.*

Many other results of this type can be obtained.

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